ISRAEL JOURNAL OF MATHEMATICS, Vol. 21, No. 4, 1975

NUMBERTHEORETICAL ENDOMORPHISMS WITH σ-FINITE INVARIANT MEASURE

BY

F. SCHWEIGER

ABSTRACT

A class of measurable transformations which serves as a model for several f-expansions is discussed. Sufficient conditions for ergodicity and the existence of a σ-finite invariant measure are given.

1. Introduction

Various models have been proposed which cover almost all known continued fraction-like expansions possessing a finite invariant measure equivalent to Lebesgue measure (Rényi [7], Schweiger [8], Fischer [4]). However, there are interesting examples of simple algorithms which do not exhibit a finite invariant measure. In this note an ergodic theory is given and some examples are worked out in detail.

2. A class of numbertheoretical endomorphisms

Let $(B, \mathcal{F}, \lambda)$ be a probability space. We consider transformations $T: B \rightarrow B$ subject to the following conditions.

(a) T is measurable and nonsingular.

(b) There is a partition $\{B(k) \mid k \in I\}$ the fibres $B(k)$ of which are measurable. The index set I is finite or countable.

(c) There is a family of measurable and nonsingular mappings $V(k): B \rightarrow B(k)$, $k \in I$, such that $V(k)T = 1_{B(k)}$ and $TV(k) = 1_B$.

(d) We define

$$V(k_1, \cdots, k_n) = V(k_1, \cdots, k_{n-1})\, V(k_n)$$

$$B(k_1, \cdots, k_n) = V(k_1, \cdots, k_{n-1})\, B(k_n)$$

and $\mathcal{Z}^{(n)}$ denotes the family of all cylinders $B(k_1, \cdots, k_n)$ of order n. Then we

Received October 21, 1974

suppose that $\mathscr{Z} = \bigcup_{n=1}^{\infty} \mathscr{Z}^{(n)}$ generates $\mathscr{F}$. In our examples this will follow from

$$\lim_{n \to 0} \left(\sup_{Z \in \mathscr{Z}^{(n)}} \operatorname{diam} Z \right) = 0 .$$

(e) We put

$$\Delta(k_1, \cdots, k_n)(x) = \frac{d\lambda V(k_1, \cdots, k_n)}{d\lambda}(x)$$

Given a constant $C \geqq 1$ we call a cylinder $B(k_1, \cdots, k_n)$ an R-cylinder if it satisfies "Rényi's condition"

(e.1)

$$\operatorname{ess} \sup_{x \in B} \Delta(k_1, \cdots, k_n)(x) \leqq C \operatorname{ess} \inf_{x \in B} \Delta(k_1, \cdots, k_n)(x) .$$

The set of all R-cylinders with constant C is denoted by $\mathscr{G}(C, T)$. The ideal case is that we can find a constant $C \geqq 1$ such that $\mathscr{G}(C, T) = \mathscr{Z}$. In this case (Lemma 4) one can show the existence of a finite invariant measure $\mu \sim \lambda$.

In the examples we want to cover the case $\mathscr{G}(C, T)$ is a proper subclass of $\mathscr{Z}$ for all $C \geqq 1$. The examples suggest to impose the following weaker condition

(e.2) There can be found a constant $C \geqq 1$ and a class $\mathscr{R}(C, T) \subseteq \mathscr{G}(C, T)$

(f.1) If $B(k_1, \cdots, k_n) \in \mathscr{R}(C, T)$ then $B(a_1, \cdots, a_s, k_1, \cdots, k_n) \in \mathscr{R}(C, T)$ for any choice of the sequence $a_1, \cdots, a_s$.

This is a kind of Markov property.

(f.2) Let

$$\mathscr{D}_n := \{B(k_1, \cdots, k_n) \in \mathscr{Z}^{(n)} \mid B(k_1, \cdots, k_s) \in \mathscr{Z} \setminus \mathscr{R}(C, T), 1 \leqq s \leqq n\},$$

then

$$\lim_{n \to \infty} \sum_{\mathscr{D}_n} \lambda(B(k_1, \cdots, k_n)) = 0.$$

Note that we do not assume

$$\sum_{n=1}^{\infty} \sum_{\mathscr{D}_n} \lambda(B(k_1, \cdots, k_n)) < \infty.$$

This condition would again imply the existence of a finite invariant measure $\mu \sim \lambda$.

In Section 3 we will prove some general results (ergodicity of T, existence of a σ-finite invariant measure $\mu \sim \lambda$). In Section 4 we will work out some examples.

3. Some general results

LEMMA 1. *For any* $B(k_1,\cdots,k_n)\in\mathscr{Z}^{(n)}\cap\mathscr{R}(C,T)$ *we have*

$$\lambda(T^{-n}E\cap B(k_1,\cdots,k_n))\geqq C^{-1}\lambda(E)\lambda(B(k_1,\cdots,k_n)).$$

PROOF.

$$\lambda(T^{-n}E\cap B(k_1,\cdots,k_n))=\int_{B(k_1,\cdots,k_n)}c_E(T^nx)d\lambda(x)$$

$$=\int_B c_E(y)\Delta(k_1,\cdots,k_n)(y)\,d\lambda(y)$$

$$\geqq C^{-1}\lambda(E)\lambda(B(k_1,\cdots,k_n)).$$

We now introduce the class

$$\mathscr{B}_n=\{B(k_1,\cdots,k_n)\in\mathscr{Z}^{(n)}\mid B(k_1,\cdots,k_n)\in\mathscr{R}(C,T),B(k_1,\cdots,k_{n-1})\in\mathscr{D}_{n-1}\}.$$

We remark that $\bigcup_{j=1}^{n}\mathscr{B}_j\cup\mathscr{D}_n$ is a disjoint covering of B.

LEMMA 2. *Any cylinder is within a set of* λ*-measure zero a disjoint union of R-cylinders.*

PROOF. We may assume $B(k_1,\cdots,k_n)\in\mathscr{Z}\setminus\mathscr{R}(C,T)$. Then we form the disjoint union representation

$$B(k_1,\cdots,k_n)=\bigcup_{t=1}^{m}\bigcup_{B(a_1,\cdots,a_t)\in\mathscr{B}_t}B(k_1,\cdots,k_n,a_1\cdots,a_t)$$

$$\cup\bigcup_{B(b_1,\cdots,b_m)\in\mathscr{D}_m}B(k_1,\cdots,k_n,b_1,\cdots,b_m).$$

Here we use (f.1). Then we have

$$\lambda\Big(\bigcup_{B(b_1,\cdots,b_m)\in\mathscr{D}_m}B(k_1,\cdots,k_n,b_1,\cdots,b_m)\Big)=\sum_{\mathscr{D}_m}\lambda(V(k_1,\cdots,k_n)B(b_1,\cdots,b_m)).$$

Using (f.2) and the fact that $\lambda(A)=0$ implies $\lambda(V(k_1,\cdots,k_n)A)=0$,

we are done.

THEOREM 1. *T is ergodic with respect to* λ.

PROOF. Let $T^{-1}E=E$, then we have

$$\lambda(E\cap Z)\geqq C^{-1}\lambda(E)\lambda(Z)$$

for any R-cylinder Z. Lemma 2 implies that this is sufficient to conclude $c_E\geqq 1$ a.e.

LEMMA 3. *T admits a σ-finite invariant measure* $\mu \sim \lambda$ *iff there exists a measurable function f such that*

$$f(x) = \sum_{k \in I} f(V(k)x)\Delta(k)(x) \qquad a.e.$$

PROOF. We take $\mu(A) = \int_A f(x) d\lambda(x)$ and consider the defining equation

$$\mu(T^{-1}A) = \sum_{k \in I} \mu(V(k)A) = \mu(A).$$

LEMMA 4. *If there is a constant D such that* $\mathcal{G}(D, T) = \mathfrak{X}$, *then T admits even a finite invariant measure* $\nu \sim \lambda$ *and* $D^{-1} \leqq f \leqq D$ *a.e.*

PROOF. In this case

$$\begin{aligned}\lambda(T^{-m}B(k_1, \cdots, k_n)) &= \sum \lambda(B(a_1, \cdots, a_m, k_1, \cdots, k_n)) \\ &= \sum \int_B \Delta(a_1, \cdots, a_m)(V(k_1, \cdots, k_n)x)\Delta(k_1, \cdots, k_n)(x)\, d\lambda(x).\end{aligned}$$

Therefore, using (e.1) which condition now applies to all cylinders, we obtain

$$D^{-1}\lambda(B(k_1, \cdots, k_n)) \leqq \lambda(T^{-m}B(k_1, \cdots, k_n)) \leqq D\lambda(B(k_1, \cdots, k_n)).$$

From known theorems in ergodic theory (see e.g. Friedman [5]) the lemma follows.

Now we define an auxiliary transformation $T^*: B \to B$ as follows:

$$T^*x = T^n x \quad \text{iff} \quad x \in B(k_1, \cdots, k_n), \qquad B(k_1, \cdots, k_n) \in \mathcal{B}_n.$$

As can be checked easily, the following lemma is true:

LEMMA 5. T^* *is a numbertheoretical endomorphism in the sense of Section 2. The fibres of the time-one-partition are the cylinders* $B(k_1, \cdots, k_n) \in \mathcal{B}_n$, $n = 1, 2, \cdots$.

Furthermore, we have

$$\mathcal{G}(C, T^*) = \mathfrak{X}^*$$

and there exists a finite T^**-invariant measure* $\mu^* \sim \lambda$.

PROOF. The assertion on the time-one-partition follows from the very definition. Using (f.1) we see that all cylinders (with respect to T^*) are

R-cylinders (with respect to T) and therefore satisfy (e.1). We can apply Lemma 4 with $D = C, T$ replaced by $T^*, \nu = \mu^*$.

If we put $\mu^*(A) = \int_A f^*(x)\, d\lambda(x)$, we note that Lemma 3 implies

$$f^*(x) = \sum_{n=1}^{\infty} \sum_{\mathcal{B}_n} f^*(V(k_1, \cdots, k_n)x)\Delta(k_1, \cdots, k_n)(x).$$

LEMMA 6. $\sum_{n=1}^{\infty} \sum_{\mathcal{D}_n} \Delta(k_1, \cdots, k_n)(x) < \infty$ *a.e.*

PROOF. Let $B(a_1, \cdots, a_m) \in \mathcal{R}(C, T)$. Then

$$\int_{B(a_1, \cdots, a_m)} \sum_{n=1} \sum_{\mathcal{D}_n} \Delta(k_1, \cdots, k_n)(x)\, d\lambda(x)$$

$$= \sum_{n=1}^{\infty} \Big(\sum_{B(k_1, \cdots, k_n) \subset \mathcal{D}_n} \lambda(B(k_1, \cdots, k_n, a_1, \cdots, a_m)) \Big)$$

$$\leqq \sum_{n=1}^{\infty} \Big(\sum_{\mathcal{D}_n} \lambda(B(k_1, \cdots, k_n) - \sum_{\mathcal{D}_{n+m}} \lambda(B(k_1, \cdots, k_n, k_{n+1}, \cdots, k_{n+m}) \Big)$$

$$= \sum_{n=1}^{m} \sum_{\mathcal{D}_n} \lambda(B(k_1, \cdots, k_n)) - \lim_{N \to \infty} \sum_{j=1}^{m} \sum_{\mathcal{D}_{n+j}} \lambda(B(k_1, \cdots, k_{N+j}))$$

$$= \sum_{n=1}^{m} \sum_{\mathcal{D}_n} \lambda(B(k_1, \cdots, k_n)).$$

Here we have used (f.2) to obtain the convergence of the majorant and the fact that (f.1) implies: Let $B(a_1, \cdots, a_m) \in \mathcal{R}(C, T)$ and $B(k_1, \cdots, k_n) \in \mathcal{D}_n$, then $B(k_1, \cdots, k_n, a_1, \cdots, a_m) \notin \mathcal{D}_{n+m}$. By Lemma 2, Lemma 6 is proved.

THEOREM 2. *T admits a σ-finite invariant measure $\mu \sim \lambda$.*

PROOF. The measurable function

$$f(x) = f^*(x) + \sum_{n=1}^{\infty} \sum_{\mathcal{D}_n} f^*(V(b_1, \cdots, b_n)x)\Delta(b_1, \cdots, b_n)(x)$$

is finite a.e. by Lemma 6. We will show that f is the density of an invariant measure:

$$\sum_{k \subset I} f(V(k)x)\Delta(k)(x) = \sum_{k \subset I} \Big[f^*(V(k)x)\Delta(k)(x) + \sum_{n=1}^{\infty} \sum_{\mathcal{D}_n} f^*(V(a_1, \cdots, a_n) \cdot V(k)x)\Delta(a_1, \cdots, a_n)(V(k)x)\Delta(k)(x) \Big]$$

$$= \sum_{k \in I} \Bigg[f^*(V(k)x)\Delta(k)(x) + \sum_{n=1}^{\infty} \sum_{B(a_1,\cdots,a_n) \in \mathcal{L}_n} f^*(V(a_1,\cdots,a_n,k)x)$$

$$\cdot \Delta(a_1,\cdots,a_n,k)(x) \Bigg]$$

$$= \sum_{n=1}^{\infty} \sum_{B(a_1,\cdots,a_{n-1},k) \subset \mathcal{B}_n} f^*(V(a_1,\cdots,a_{n-1},k))\Delta(a_1,\cdots,a_{n-1},k)(x)$$

$$+ \sum_{n=1}^{\infty} \sum_{B(a_1,\cdots,a_{n-1},k) \in \mathcal{L}_n} f^*(V(a_1,\cdots,a_{n-1},k))\Delta(a_1,\cdots,a_{n-1},k)(x)$$

$$= f^*(x) + \sum_{n=1}^{\infty} \sum_{\mathcal{L}_n} f^*(V(a_1,\cdots,a_{n-1},k))\Delta(a_1,\cdots,a_{n-1},k)(x) = f(x).$$

Note that we have used $B(a_1,\cdots,a_{n-1},k) \in \mathcal{B}_n$ iff $B(a_1,\cdots,a_{n-1}) \in \mathcal{D}_{n-1}$ and $B(a_1,\cdots,a_{n-1},k) \in \mathcal{R}(C,T)$.

REMARK. f is integrable (and μ is a finite measure iff $\sum_{n=1}^{\infty} \sum_{\mathcal{L}_n} \lambda(B(k_1,\cdots,k_n)) < \infty$.

4. Examples

All examples given are on the probability space $B = [0,1] \bmod 0$, $\mathcal{F} = \sigma$-algebra of Borel sets, λ Lebesgue measure. We first prove

LEMMA 7. *Suppose that all $V(k)$ are continuously differentiable and there is a point $y \in [0,1]$ such that $Ty = y$ and $T'y = 1$, then for every $C \geqq 1$ the class $\mathcal{G}(C,T)$ is a proper subclass of $\mathcal{L}$.*

PROOF. From $Ty = y$ we see that y has a purely periodic expansion $k_s(y) = b$, $s = 1,2,\cdots$. Therefore for every $n \geqq 1$ by the chain rule we obtain

$$\Delta(\underbrace{b,\cdots,b}_{n})(y) = \Delta(\underbrace{b,\cdots,b}_{n})(T^n y) = \left[\frac{dT^n}{dx}(y) \right]^{-1} = 1.$$

Hence,

$$\sup_{x \in B} \Delta(\underbrace{b,\cdots,b}_{n})(x) \geqq 1.$$

On the other hand,

$$\lambda(B(\underbrace{b,\cdots,b}_{n})) = \int_B \Delta(\underbrace{b,\cdots,b}_{n})(x)d\lambda(x).$$

Therefore

$$\inf_{x \in B} \Delta(b,\cdots,b)(x) \leqq \varepsilon(n),$$

where $\lim_{n\to\infty} \varepsilon(n) = 0$.

Corollary. *In such a point* y *we clearly have* $f(y) = +\infty$.

Proof. Insert y in the formula given by Theorem 2 and observe $V(b,\cdots,b)y = y$.

Now we discuss several examples.

(1) This example is due to Rényi (Adler [1]). It is also discussed in Rudolfer [8].

$$I = \{0,1,2,\cdots,.\}$$

$$B(k) = \left[\frac{k}{k+1}, \frac{k+1}{k+2}\right]$$

$$V(k)(x) = \frac{x+k}{x+k+1}$$

$$Tx = \frac{x}{1-x} \bmod 1$$

$$\Delta(k_1,\cdots,k_s)(x) = \frac{1}{(C_s + D_s x)^2}$$

$$C_1 = k_1 + 1, \qquad D_1 = 1$$

$$C_{s+1} = (C_s + D_s)k_{s+1} + C_s, \qquad D_{s+1} = C_s + D_s.$$

Therefore $C_s \geqq D_s$ iff $k_s \geqq 1$.
We take $C = 4$ and we take

$$\mathscr{R}(4,T) = \{B(k_1,\cdots,k_s) \mid k_s \geqq 1\}$$

$$\mathscr{D}_s = \{B(\underbrace{0,\cdots,0}_{s})\}.$$

We remark that $T0 = 0$ and $T'0 = 1$. In fact,

$$\Delta(\underbrace{0,\cdots,0}_{s})(x) = \frac{1}{(1+sx)^2}$$

$$\lambda(B(\underbrace{0,\cdots,0}_{s})) = \frac{1}{1+s}.$$

The density of the invariant measure is known:

$$f(x) = \frac{1}{x}.$$

Now take

$$E = B(\underbrace{0,0,\cdots,0}_{s}) = \left[0, \frac{1}{1+s}\right].$$

Since the indicator function of $[0,1]\setminus E$ is integrable with respect to μ, the ergodic theorem implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} c_E(T^j x) = 1.$$

This result states that the frequency of a block $\underbrace{0\cdots 0}_{s}$ in the sequence of digits is a.e. one.

Put $\Psi(x) = 1 - x$, then the transformation $\Psi T \Psi$ gives a continued fraction-like algorithm which has been useful in algebraic geometry (Hirzebruch [6], Cohn [3]).

(2) This algorithm is mentioned in Adler [1].

$$I = \{0,1\}$$

$$B(k) = \left[\frac{k}{2}, \frac{k+1}{2}\right]$$

$$V(k)x = \frac{1}{2\pi} \arctan (2 \tan \pi x) + \frac{k}{2}$$

$$Tx = \frac{1}{\pi} \arctan \frac{\tan 2\pi x}{2}.$$

A direct calculation of $\Delta(k_1,\cdots,k_s)$ seems to be complicated, but Fischer was able to prove

LEMMA 8 (Fischer). *Rényi's condition* (e.1) *is satisfied for the class*

$$\mathcal{R}(e^{6\pi}, T) = \{B(k_1,\cdots,k_s) \mid k_{s-1} = 0, k_s = 1 \text{ or } k_{s-1} = 1, k_s = 0\}.$$

PROOF. We note

$$T'x = \frac{4}{1 + 3\cos^2 2\pi x}, \quad T''x = \frac{24\pi \sin 4\pi x}{(1 + 3\cos^2 2\pi x)^2}.$$

Therefore,

$$|T''z / T'z| \leqq 6\pi \quad \text{for all} \quad z \in B.$$

Clearly

$$\sup_{x,y\in B}\left|\frac{\Delta(k_1,\cdots,k_s)(x)}{\Delta(k_1,\cdots,k_s)(y)}\right| = \sup_{n,v\in B(k_1,\cdots,k_s)}\left|\frac{(T^s)'n}{(T^s)'v}\right|.$$

Then we estimate

$$\log\left|\frac{(T^s)'n}{(T^s)'v}\right| \sum_{i=0}^{s-1}|\log T'(T^i n)-\log T'(T^i v)|$$

$$\leqq 6\pi\sum_{i=0}^{s-1}|T^i n - T^i v|$$

$$\leqq 6\pi\sum_{i=0}^{s-1}\lambda(B(k_{i+1},\cdots,k_s)).$$

We will only discuss the case $k_{s-1}=0, k_s=1$. Then we will show

$$(*)\qquad \lambda(B(a_1,\cdots,a_n,0,1))\leqq\lambda(B(\underbrace{0,\cdots,0}_{n},0,1))$$

for any choice of $(a_1,\cdots,a_n)$.

If we take this result as granted, we conclude

$$\sum_{i=0}^{s-2}\lambda(B(k_{i+1},\cdots,k_{s-2},0,1))+\lambda(B(1))$$

$$\leqq\sum_{i=0}^{s-1}\lambda(B(\underbrace{0,\cdots,0}_{i},1))\leqq 1.$$

Now we will prove $(*)$ by induction on n. The case $n=0$ is clear. Using

$$\Delta(k)(x)=\frac{2}{5-3\cos 2\pi x}$$

$$\cos 2\pi V(0)x = -\cos 2\pi V(1)x,$$

one verifies (by induction on t)

$$\Delta(k)(V(\underbrace{0,\cdots,0}_{t},0,1)1)=\Delta(k)(V(\underbrace{1,\cdots,1}_{t},0,1)1).$$

Now let us assume $(a_2,\cdots,a_n)\neq(0,\cdots,0)$. Then we have

$$V(\underbrace{0,\cdots,0}_{n-1},0,1)1\leqq V(a_2,\cdots,a_n,0,1)x$$

$$\leqq V(\underbrace{1,\cdots,1}_{n-1},0,1)1.$$

Now we calculate

$$\begin{aligned}
&\lambda(B(a_1,\cdots,a_n,0,1))\\
&=\int_0^1 \Delta(a_1,\cdots,a_n,0,1)(x)\,d\lambda(x)\\
&=\int_0^1 \Delta(a_1)(V(a_2,\cdots,a_n,0,1)x)\Delta(a_2,\cdots,a_n,0,1)(x)\,d\lambda(x)\\
&\leqq \Delta(a_1)(V(\underbrace{0,\cdots,0}_{n-1},0,1)\lambda(B(a_2,\cdots,a_n,0,1))\\
&\leqq \Delta(a_1)(V(\underbrace{0,\cdots,0}_{n-1},0,1)1)\lambda(B(\underbrace{0,\cdots,0}_{n-1},0,1))\\
&\leqq \int_0^1 \Delta(a_1)(V(\underbrace{0,\cdots,0}_{n-1},0,1)1)x)\Delta(\underbrace{0,\cdots,0}_{n-1},0,1)(x)\,d\lambda(x)\\
&=\lambda(B(\underbrace{0,\cdots,0}_{n},0,1)).
\end{aligned}$$

Here we used that $\Delta(k)$ is independent of k. Similarly one calculates

$$\lambda(B(1,\underbrace{0,\cdots,0}_{n-1},0,1))=\lambda(B(0,\underbrace{0,\cdots,0}_{n-1},0,1)).$$

Now we have seen that we can choose $C=e^{6\pi}$ and $\mathcal{R}(e^{6\pi},T)$ according to the preceding lemma.

$$\mathcal{D}_s=\{B(\underbrace{0,\cdots,0}_{s}),B(\underbrace{1,\cdots,1}_{s})\}.$$

In this case $T0=0$, $T1=1$ and $T'0=T'1=1$. Using

$$\cos^2 2\pi V(k)x=\frac{1+\cos 2\pi x}{5-3\cos 2\pi x},$$

one can prove

$$\lim_{n\to\infty}\cos 2\pi V(\underbrace{0,\cdots,0}_{n})x=\lim_{n\to\infty}\cos 2\pi V(\underbrace{1,\cdots,1}_{n})x=1.$$

From this condition (f.2) follows. The density of the invariant measure is known:

$$f(x)=\frac{1}{1-\cos 2\pi x}.$$

NOTE. The transformation $S:\mathcal{R}\to\mathcal{R}$ defined by $Sx = x - 1/x$ a.e. preserves Lebesgue measure. We define

$$\psi:[0,1]\to\mathcal{R}, \qquad x\to\tan\left(x\pi-\frac{\pi}{2}\right) \text{ a.e.,}$$

then easily that

$$S\psi = \psi T$$

and

$$\lambda(\psi E) = \int_E \frac{2}{1-\cos 2\pi x}\,dx.$$

Hence we can deduce from Theorem 1 the main theorem of the paper by Adler and Weiss [2]: S is ergodic.

REFERENCES

1. R. L. Adler, *F-expansions revisted*, in *Recent Advances in Topological Dynamics*, Lecture Notes in Mathematics **318** (1973), 1–5.

2. R. L. Adler and B. Weiss, *The ergodic infinite measure preserving transformation of Boole*, Israel J. Math. **16** (1973), 263–278.

3. H. Cohn, *Support polygons and the resolution of modular functional singularities*, Acta Arith. **24** (1973), 261–278.

4. R. Fischer, *Ergodische Theorie von Ziffernentwicklungen in Wahrscheinlichkeitsräumen*, Math. Z. **128** (1972), 217–230.

5. N. A. Friedman, *Introduction to Ergodic Theory*, Van Nostrand-Reinhold Co., 1970.

6. F. Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei komplexen Veränderlichen*, Math. Ann. **126** (1953), 1–22.

7. A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

8. S. Rudolfer, *Ergodic properties of linear fractional transformations* mod *one*, Proc. London Math. Soc. (3) **23** (1971), 515–531.

9. F. Schweiger, *Metrische Theorie einer Klasse zahlentheoretischer Transformationen*, Acta Arith. **15** (1968), 1–18 and **16** (1969), 217–219.

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN, U.S.A.